Frequency Conversion by Third Class Conductor
and
Mechanism of the Arcing Ground and other Cumulative Surges

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Review of the Subject.—*In high-voltage power circuits, such as transmission lines and the high-voltage coils of large power transformers, not infrequently disturbances are observed of a frequency differing from, and usually very much higher than that of the power supply, and differing from the typical transient of energy readjustment, in that they do not gradually die out, but increase in intensity until either destruction occurs, or they finally limit themselves. Such cumulative oscillations or arcing grounds derive their energy from the machine power of the system, and so constitute a frequency transformation, of which the mechanism has been little understood.*

Physically they may be derived from the typical condenser discharge by the conception of a **negative resistance,** *in combination with a* **source of power,** *which supplies the energy given out by the negative resistance.*

Attention is drawn to a class of conductors—to which arcs and gas discharges belong—the so-called "third-class conductors," in which the voltage decreases with increase of current, and it is shown that these conductors can be considered as a combination of a negative resistance with a source of power, and as such are capable of transforming the low machine frequency into a high oscillation frequency of alternating currents, and their presence in an electric system thereby may produce cumulative oscillations.

The general equations are then derived of a system comprising a third-class conductor shunted by an inductive circuit containing capacity, and supplied with voltage over an inductive circuit from an alternating low-frequency source, and it is shown that in such a system currents and voltages of two distinct frequencies may continuously exist, of which the one is the machine frequency, the other a high oscillation frequency. It is further shown that the voltage of the latter is limited only by the resistance of the oscillating circuit, and in low-resistance circuits may build up to very high values. Furthermore, the high oscillation frequency is essentially limited to the circuit shunting the third-class conductor and but little of it enters the supply circuit, while the supply frequency enters the shunt circuit to a limited extent only, and both frequencies are superimposed in the third-class conductor as the frequency converter.

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A. Physical Explanation

I. The Third Class Conductor

In a first class or metallic conductor, the voltage increases slightly more than proportional to the current, due to a positive temperature coefficient of resistance.

In a second class or electrolytic conductor, the voltage increases slightly less than proportional to the current, due to a slightly negative temperature coefficient of resistance.

As third class conductor may be defined a conductor in which, at least within a certain range of current, the voltage *decreases* with *increase* of current. Third class conductors comprise different types, such as

(a) Electronic or vacuum, gas or Geissler tube (spark) and vapor or arc conduction. In these the decrease of voltage with increase of current is due to the change of the conducting path by the current.

(b) Most of the so-called insulators probably are third class conductors. In these the decrease of voltage with increase of current is a temperature effect. That is, the negative temperature coefficient of resistance is so large that—at least in a certain range—the increase of the conductor temperature with increasing current decreases the resistance more than the current increases.

In Fig. 1, *I* gives the volt-ampere characteristic of a metallic conductor, *II* of an electrolytic conductor, and *III* of an arc as third class conductor. Fig. 2, gives the volt-ampere characteristic of a pyroelectric

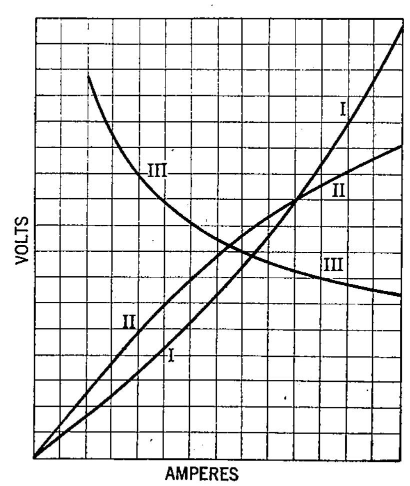

Fig. 1—Volt-Ampere Characteristics of Different Conductor Types
1. First class or metallic conductor
2. Second class or electrolytic conductor
3. Arc as third class conductor

conductor, a Nernst lamp glower, which in the range above 0.02×10^{-3} ampere is a third class conductor

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(plotted for convenience with the fourth root of current and voltage as coordinates).

Such a volt-ampere characteristic of a third class conductor can be considered as a combination of a constant e. m. f. E_1 and an effective negative resistance r_1:

$$e = E_1 - r_1\, i$$

The effective negative resistance r_1 decreases with increase of current, as $r_1\, i$ must always remain smaller than E_1.

II. Time Lag

If the current in the third class conductor varies periodically, between i_1 and i_2, the voltage also will vary periodically, between e_1 and e_2, and if the variation is slow enough so that at every value of current stationary condition is reached, the variation of voltage is inverse to that of current. That is, if i_1 is the minimum value of current, the corresponding value of voltage e_1 is the maximum value, and inversely. If however, the variation is sufficiently rapid, a lag of the voltage occurs behind the current, that is, the maximum value of voltage is not reached at the minimum value of current, but a little later, and so the minimum value of the voltage is reached a little later than the maximum value of current, and correspondingly, the amplitude of the variation is reduced, the more so, the more rapid the pulsation.

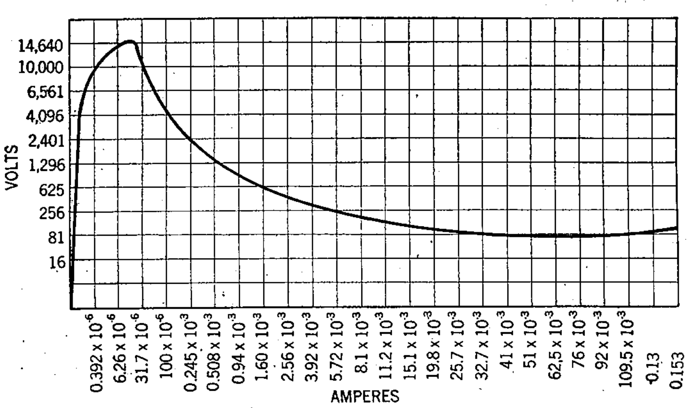

Fig. 2—Volt-Ampere Characteristics of Nernst Lamp Glower
As Third Class Conductor

This time lag of the third class conductor, and the reduction of amplitude of variation resulting from it, depends on the nature of the conductor. It is extremely small in ionic conduction, but may be very considerable where the variation of the effective resistance of the third class conductor is a temperature effect.

As first approximation, the periodic variations of current and voltage in a third class conductor thus may be expressed by:

$$i = I_0 - I \sin 2\,\pi\, f\, t$$
$$e = E_0 + E \sin (2\,\pi\, f\, t - \varphi)$$

where the lag angle φ increases with increasing frequency f of the pulsation. It is negligible for very low frequency, and becomes 90 deg., and the amplitude of the resistance variation therefore negligible, for very high frequency.

In ionic conduction, φ is still small at ratio frequencies; in the Nernst lamp glower (temperature effect) φ is already practically 90 deg. at 60 cycles.

The volt-ampere characteristic of the third class conductor under rapidly varying current (its "transient characteristic") thus differs more or less from its "permanent" characteristic as shown in Figs. 1 and 2.

III. The Condenser Discharge

If a condenser of capacity C discharges through an inductance L and (constant) ohmic resistance r, and the latter is below the critical value, the discharge current is oscillatory, that is, consists of successively alternating half waves of constant frequency and uniformly decreasing amplitude. The decrease of the current, or the "attenuation" is given by the exponential factor:

$$A = \epsilon^{-\frac{r}{2L}t}$$

If then the discharge circuit of the condenser contains, besides the ohmic resistance r, a third class conductor of the effective negative resistance r_1, the attenuation of the condenser discharge is:

$$A = \epsilon^{-\frac{r - r_1}{2L}t}$$

Thus, if the effective negative resistance r_1 is greater than the ohmic resistance r, the exponent

$$-\frac{r - r_1}{2\,L}\,t$$

is positive, and A increases with the time t, thus is not an attenuation but an accumulation. The amplitudes of successive half waves of the condenser discharge current then progressively increase, that is, we get a cumulative surge.

As seen in *I*, the effective resistance r_0 of the third class conductor decreases with increase of current. In such a cumulative surge produced by the presence of a third class conductor in the inductive condenser discharge circuit, the successive half waves of current will progressively increase, until by the increase of current the effective negative resistance r_1 has decreased to equality with the ohmic resistance r, and the exponent of the exponential term A has become zero, $A = 1$, and the successive half waves of the condenser discharge current become equal, that is, an alternating current results.

Thus the final result of a condenser discharge through a third class conductor of sufficiently high effective negative resistance is an alternating current of a frequency determined by the circuit constants.

IV. Energy Relations

A current i through the effective negative resistance $- r_1$ consumes a voltage $- r_1\, i$ which is in phase with the current, thus represents electric power generation.

Hence an effective negative resistance can exist independently only in the presence of some other source of energy, supplying the power (mechanical drive for instance). Thus an electric generator may be considered as an effective negative resistance. The most typical negative resistance is the induction machine driven above synchronism, since (below saturation) it generates a voltage proportional to the current, thus is a constant negative resistance.

As seen in *I*, a third class conductor may be considered as the combination of a counter e. m. f. E_1, and an effective negative resistance $- r_1$: $e = E_1 - r_1 i$. As $r_1 i$ must always remain smaller than E_1, therefore the power generated by the effective negative resistance: $- r_1 i^2$ always is less than the power consumed by the counter e. m. f. E_1: $E_1 i$ and the counter e. m. f. E_1 thus abstracts from the electric circuit the power which is returned by the effective negative resistance r_1.

In the condenser discharge through a third class conductor, the current wave builds up cumulatively by the effective negative resistance r_1, but the condenser voltage is lowered discontinuously by the counter e. m. f. E_1, at every reversal of current, and with it the current wave is lowered, and as E_1 is greater than $r_1 i$, the discharge oscillation gradually dies out, (though by a different law than in the standard condenser discharge equation,) as discussed in a previous paper.[1]

If however the counter e. m. f. E_1 of the third class conductor is supplied by some outside source of electric power, and therefore does not abstract energy from the condenser discharge, then the cumulative effect of the negative resistance of the third class conductor r_1 on the condenser discharge current i would continue until limited by the decrease of r_1 with increasing i, and an alternation results.

Now the importance of this phenomenon is, that the character of the supply voltage giving the counter e. m. f. E_1 has no necessary relation to the frequency of the condenser discharge; the latter is determined by the values of the capacity and inductance, and may be a high or very high frequency, while E_1 may be of machine frequency, 25 or 60 cycles, or even continuous voltage. The power supplied by the effective negative resistance to the high-frequency condenser discharge is derived from the counter e. m. f. E_1 and the latter is fed from the low-frequency machine power. We therefore have here a frequency transformation giving a steady power supply to the high-frequency oscillation, so that the latter are not any more limited energy transients, but unlimited power permanents, of corresponding destructiveness. As such they have been frequently observed in electric power systems, as arcing grounds in transmission lines, and as high-frequency cumulative surges in the high-voltage coils of large power transformers.

As the exact mechanism of this frequency transformation from the low machine power frequency to the high frequency of the cumulative surge is not generally familiar, the following may be of interest, though a similar problem has been studied in a different manner in radio engineering in the theory of the vacuum tube as oscillator.

V. Frequency

Suppose a condenser of capacity C discharges through an inductance L, an ohmic resistance r and a third class conductor N, and upon this third class conductor, a constant alternating voltage e_0 is impressed through a supply circuit of resistance r_0 and inductance L_1, as shown diagrammatically in Fig. 3. We can consider the arrangement as a divided circuit consisting of a third class conductor N in shunt to the circuit C, L, r, and energized over the circuit L_0, r_0 by voltage e_0.

We can assume that r is sufficiently small to make the condenser discharge oscillatory, and that the frequency f of this oscillation is high compared with the frequency f_0 of the supply voltage e_0.

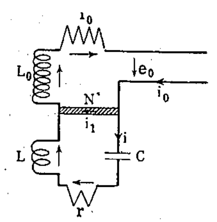

Fig. 3—Circuit Diagram of Frequency Transformation by Third Class Conductor

N: Third class conductor
C = Capacity, L = inductance, r = resistance of shunt circuit
L_0 = Inductance, r_0 = resistance of supply circuit
e_0 = low-frequency supply voltage

Denoting then the currents in the two branch circuits by i and i_1, the voltage by e_1, the differential equations can be written as those of a circuit containing resistance, inductance and capacity.

As seen therefore, if the effective negative resistance of the third class conductor is sufficiently high, the oscillatory starting transient of the condenser C does not die out, but cumulatively increases in amplitude, until limited by the decreasing negative resistance, and so reaches an alternating sine wave as final value, i, i_1 and e_1 then must consist of sine waves.

The differential equations are therefore integrated by representing i, i_1 and e_1 by summation of sine waves.

$$\Sigma B \sin (q t - \beta)$$

substituting these into the differential equations, and from the identities thus produced, calculate the constants B, β etc.

One of the terms must be of the supply frequency $q = 2 \pi f_0$, and the amplitude and the phase angle of its currents and voltages are given by the terminal condition, that the supply voltage is e_0.

This term merely gives the permanent low-frequency alternating current and voltage distribution in the divided circuit Fig. 3.

1. Condenser Discharge Through a Gas Circuit A. I. E. E., Feb. 1922.

For all other terms in the solution of the differential equation, the impressed e. m. f. equals zero. This condition gives an equation, which determines the frequency $q = 2\pi f$, of the term.

It follows therefore, that besides the low-frequency term of the supply frequency f_0, only one second term can exist, in the equations of current and voltage, of an oscillation frequency f of the condenser discharge circuit.

The frequency of local oscillation f, which from the viewpoint of frequency transformation may be called the "secondary frequency" depends on the constants of the circuit, and therefore has no numerical relation to the supply or "primary" frequency, is not a multiple or a higher harmonic of it, as in most other frequency transformations.

VI. Amplitude

If an alternating supply voltage is impressed upon a third class conductor contained in an inductive condenser discharge circuit, the transient condenser oscillation may build up to a permanent high-frequency alternation. Current and voltage then contain two terms. The first or low-frequency one is that due to the impressed e. m. f. In the second the frequency is determined by the circuit constants, but the amplitude thus far left indeterminate. The amplitude of the high-frequency current and voltage thus increases, until finally limited by the decreasing effective negative resistance of the third class conductor, and by the supply voltage.

As the effective negative resistance of the third class conductor must be greater than the ohmic resistance of the circuit, to give a cumulative oscillation, and as this effective negative resistance decreases with increase of current, the limit, to which the high-frequency alternating current can build up, is that value, at which the effective negative resistance of the third class conductor had decreased to equality with the ohmic resistance of the circuit, as discussed in III.

As the only source of power is the low-frequency supply by the impressed voltage e_0, this must always be positive. That is, in the voltage e_1 and the current i_1 of the third class conductor, the high-frequency term —which absorbs power—if of opposite sign of the low-frequency term—which supplies the power—must always be less than the low-frequency term, to maintain positive power. This means, the maximum value of the high-frequency term must be less than the instantaneous value of the low-frequency term at the moment when the maximum value of the high-frequency term occurs. Therefore the amplitude of the high-frequency term is not constant, but periodically rises and falls, with the frequency of the low-frequency term and the latter thus is the envelope of the high-frequency term, as illustrated in Fig. 4.

The difference between this type of condenser discharge, and the standard form is that in the latter the envelope of the periodic discharge current is exponential, and the current thus gradually dies out, as transient, while in the present case, with power supply through a third class conductor, the envelope of the condenser discharge current is a low-frequency sine wave, and the discharge current thus periodically rises and falls, but persists as permanent.

Such a periodically rising and falling alternating wave can be considered as the superposition of two waves, differing in frequency from each other by the frequency of the beats, that is, in the present case, twice the frequency of the supply voltage; hence of the frequency $f - f_0$ and $f + f_0$. However, as at very low currents the third class conductor usually ceases to represent an effective negative resistance, the periodic high-frequency oscillation usually dies out in a short range at the reversal of the low-frequency wave, and then starts again at the rise of the latter, as indicated diagrammatically in Fig. 4, and the phases of the groups of the high-frequency waves in the successive low-frequency half waves therefore are independent of each other. The resolution of the high-frequency waves into two components of different frequency therefore has no physical significance.

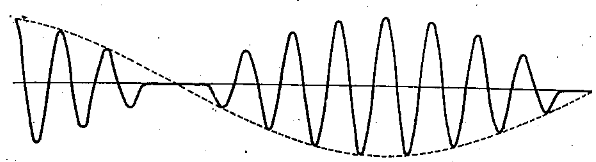

Fig. 4—High-Frequency Secondary and Low-Frequency Primary Wave of Frequency Transformation by Third Class Conductor

As the maximum value of the high-frequency wave is limited to the instantaneous value of the low-frequency wave, the mean square of the maximum value of the high-frequency wave is limited to the mean square of the instantaneous value, and therefore to half the square of the maximum value of the low-frequency wave. This means, that the power of the high-frequency alternation is limited to half the power of the low-frequency wave; in other words, under the conditions herein considered, the efficiency of the frequency transformation cannot exceed 50 per cent.

If the secondary frequency f is much higher than the primary frequency, very little low-frequency enters the supply circuit, since it is kept out by the high capacity reactance, at the low-frequency of the condenser C, and very little high-frequency current enters the supply circuit, since it is kept out by the high inductive reactance, at the high frequency, of L_0. Thus the capacity shunt is essentially a high-frequency circuit, the supply a low-frequency circuit, and both frequencies concur in the third class conductor, in which the transformation takes place.

B. Mathematical Calculation

I. General Equation

Let a third class conductor be shunted by a circuit of capacity C, inductance L, and ohmic resistance r and supplied by a constant alternating e. m. f. of machine frequency, over a resistance r_0 and inductance L_0.

We may assume that r is below the critical value, so that the current in C, L, r is oscillatory, and that the frequency of oscillation of this circuit is high compared with the machine frequency, and that L_0 is large compared with L.

Let the alternating supply e. m. f. be a sine wave:

$$e_0 = E_0 \sin (q_0 t - \alpha) \qquad (1)$$

where:

$q_0 = 2 \pi f_0 t,$

and,

f_0 = frequency of power supply.

Let:

e_1 = voltage at the terminals of the third class conductor

i_1 = current in the third class conductor

i = current in the circuit C, L, r shunting the third class conductor

e' = voltage at the terminals of the condenser C.

e_0, e_1, i_1, i and e' being counted in the direction as indicated by the arrows in Fig. 3.

It is then, at the condenser C:

$$i = C \frac{d e'}{d t} \qquad (2)$$

It is, in the closed circuit between the third class conductor and the capacity inductance shunt:

$$e_1 = e' + r i + L \frac{d i}{d t} \qquad (3)$$

Hence, differentiating (3) and substituting (2) therein, gives:

$$\frac{d e_1}{d t} = i/C + r \frac{d i}{d t} + L \frac{d i^2}{d t^2} \qquad (4)$$

It is, in the supply circuit:

$$e_0 = e_1 + r_0 (i_1 + i) + L_0 \frac{d (i_1 + i)}{d t} \qquad (5)$$

and the voltage across the condenser terminal is, by (3):

$$e' = e_1 - r i - L \frac{d i}{d t}$$

Assuming that the circuits have been closed sufficiently long to reach stationary conditions, that is, that the starting transient has passed and any cumulative oscillation which may occur has built up to its final value.

e_1, i_1 and i then must be periodic functions, and as such may be represented by a sume of sine waves:

$$\left.\begin{aligned} e_1 &= \Sigma E_1 \sin (q t - \beta) \\ i_1 &= \Sigma I_1 \sin (q t - \gamma) \\ i &= \Sigma I \sin (q t - \omega) \end{aligned}\right\} \qquad (7)$$

Substituting (7) into (3) then must give an identity, that is, the coefficient of $\cos q t$ and $\sin q t$ must individually disappear. This gives:

$$\left.\begin{aligned} E_1 \cos \beta &= I (r \cos \omega - R \sin \omega) \\ E_1 \sin \beta &= I (r \sin \omega + R \cos \omega) \end{aligned}\right\}$$

where

$$R = \frac{1}{q C} - q L \qquad (8)$$

Herefrom follows:

$$E_1 = z I \qquad (9)$$

and

$$\tan \omega = \frac{r \sin \beta - R \cos \beta}{r \cos \beta + R \sin \beta} \qquad (10)$$

where:

$$z = \sqrt{R^2 + r^2} \qquad (11)$$

Substituting (1) and (7) into (5) then gives:

$$E_0 \sin (q_0 t - \alpha) = \Sigma E_1 \sin (q t - \beta) + \Sigma z_0 \{I_1 \cos (q t - \gamma - \zeta) + I \cos (q t - \omega - \zeta)\} \qquad (12)$$

where:

$$z_0 = \sqrt{q^2 L_0^2 + r_0^2} \qquad (13)$$

$$\tan \zeta = \frac{r_0}{q L_0} \qquad (14)$$

II. Low Frequency Terms

(12) also must be an identity. That is, the coefficients of $\cos q t$ and $\sin q t$ must individually vanish. This gives, for: $q = q_0$; denoting the terms by the index O:

$$\left.\begin{aligned} E_0 \sin \alpha &= E_1 \sin \beta_0 - z_0 \{I_1 \cos (\gamma_0 + \zeta_0) + I \cos (\omega_0 + \zeta_0)\} \\ E_0 \cos \alpha &= E_1 \cos \beta_0 + z_0 \{I_1 \sin (\gamma_0 + \zeta_0) + I \cos (\omega_0 + \zeta_0)\} \end{aligned}\right\} \qquad (15)$$

Let:

$$r_1 = E_1/I_1 \qquad (16)$$

= effective resistance of the third class conductor.

Substituting (**16**) and (**9**) into (**15**), gives two equations in I and α, which resolved give:

$$I = \frac{E_0}{\sqrt{\left[z_0\left\{\frac{z_0}{r_1}\cos(\gamma_0+\zeta_0)+\cos(\omega_0+\zeta_0)\right\}-z_0\sin\beta_0\right]^2 + \left[z_0\left\{\frac{z_0}{r_1}\sin(\gamma_0+\zeta_0)+\sin(\omega_0+\zeta_0)\right\}+z_0\cos\beta_0\right]^2}} \quad (17)$$

$$\tan\alpha = \frac{z_0\sin\beta_0 - z_0\left\{\frac{z_0}{r_1}\cos(\gamma_0+\zeta_0)+\cos(\omega_0+\zeta_0)\right\}}{z_0\cos\beta_0 + z_0\left\{\frac{z_0}{r_1}\sin(\gamma_0+\zeta_0)+\cos(\omega_0+\zeta_0)\right\}} \quad (18)$$

These expressions can be very much simplified by neglecting terms of secondary order, and choosing the phase of the current in the third class conductor as base line, that is, $\gamma_0 = 0$.

For the low frequency $q = q_0$, the time lag of the third class conductor can be neglected, giving $\beta_0 = \gamma_0 = 0$. By (**10**), this gives

$$\tan\omega_0 = -R_0/r \quad (19)$$

Neglecting the resistance r_0 against the reactance of the supply circuit $x_0 = q_0 L_0$, gives by (**14**) $\zeta_0 = 0$, and by (**13**):

$$z_0 = q_0 L_0 = x_0 \quad (20)$$

For the low-frequency $q = q_0$, $q_0 L$ is negligible against $\frac{1}{q_0 C}$, and, denoting the latter, the capacity reactance, by

$$k = \frac{1}{q_0 C} \quad (21)$$

it is by (**8**), (**11**), (**9**) and (**19**):

$$\left.\begin{aligned} R_0 &= k \\ Z_0 &= k \\ E_1^0 &= k I \\ \tan\omega_0 &= -k/r \end{aligned}\right\} \quad (22)$$

Thus approximately:

$$\omega_0 = -90 \text{ deg.}$$

and we now get from (**17**) and (**18**):

$$I^0 = \frac{E_0}{k\sqrt{\left(\frac{x_0}{r_1}\right)^2 + \left(1-\frac{x_0}{k}\right)^2}} = \frac{E_0}{k\sqrt{\left(\frac{x_0}{r_1}\right)^2+1}} \quad (23)$$

$$\tan\alpha = -\frac{x_0}{r_1}\left(1+\frac{x_0}{k}\right) = -\frac{x_0}{r_1} \quad (24)$$

and herefrom the approximate values of E^0, I_1^0, etc.

III. High-Frequency Terms

In the identity (**12**), it is for:

$$q \neq q_0$$

$$\left.\begin{aligned} 0 &= E_1\sin\beta - z_0\{I_1\cos(\gamma+\zeta) + I\cos(\omega+\zeta)\} \\ 0 &= E_1\cos\beta + z_0\{I_1\sin(\gamma+\zeta) + I\sin(\omega+\zeta)\} \end{aligned}\right\} \quad (25)$$

Substituting (**9**) into (**25**) and rearranging; gives:

$$\left.\begin{aligned} z_0 I_1\cos(\gamma+\zeta) &= I\{z\sin\beta - z_0\cos(\omega+\zeta)\} \\ z_0 I_1\sin(\gamma+\zeta) &= -I\{z\cos\beta + z_0\sin(\omega+\zeta)\} \end{aligned}\right\} \quad (26)$$

Thus

$$\tan(\gamma+\zeta) = \frac{z_0\sin(\omega+\zeta) + z\cos\beta}{z_0\cos(\omega+\zeta) - z\sin\beta} \quad (27)$$

If L_0 is large compared with L, then z is small compared with z_0, and can be neglected against it, and it thus is:

$$\tan(\gamma+\zeta) = \tan(\omega+\zeta) \quad (28)$$

$$\gamma = \omega \quad (29)$$

Substituting (**29**) into (**10**) and rearranging, gives:

$$\begin{aligned} R/r &= \tan(\beta-\gamma) \\ &= \tan\delta \end{aligned} \quad (30)$$

where

$$\delta = \beta - \gamma \quad (31)$$

is the phase difference between current and voltage in the third class conductor, or the time lag angle of the third class conductor.

Substituting (**8**) into (**30**), and resolving, gives:

$$q = -\frac{r}{2L}\tan\delta + \sqrt{\frac{1}{LC} + \frac{r^2}{4L^2}\tan^2\delta} \quad (32)$$

[If the angle of time lag of the third class conductor vanishes: $\delta = \beta - \gamma = 0$, that is, current and voltage are in phase with each other, then

$$q = \frac{1}{\sqrt{LC}}$$

the same is the case, if $r = 0$, that is, the ohmic resistance of the condenser inductance shunt is negligible.]

As seen, only two terms can exist in the equation (**6**) the one with q_0, of the machine frequency, due to the impressed voltage e_0, and the one with q, (**32**), the frequency of oscillation of the inductive condenser circuit.

From (**30**), (**11**) and (**9**) follows:

$$\left.\begin{aligned} R &= r\tan\delta \\ z &= \frac{r}{\cos\delta} \\ I &= \frac{E_1\cos\delta}{r} \end{aligned}\right\} \quad (33)$$

and by substituting into (**26**), and neglecting r against z_0, gives:

$$I_1 = -I = -\frac{E_1\cos\delta}{r} \quad (34)$$

IV. Amplitude

Combining the low and high-frequency terms then gives the total expressions of current and voltage:

$$
\left.\begin{aligned}
e_0 &= E_0 \sin (q_0 t + \alpha_0) \\
\tan \alpha_0 &= + \frac{x_0}{r_1} \\
e_1 &= \frac{E_0}{\sqrt{1 + \left(\frac{x_0}{r_1}\right)^2}} \sin q_0 t + E_1 \sin (q t - \delta) \\
q &= - \frac{r}{2 L} \tan \delta + \sqrt{\frac{1}{L C} + \frac{r^2}{4 L^2} \tan^2 \delta} \\
i_1 &= \frac{E_0}{\sqrt{r_1^2 + x_0^2}} \sin q_0 t - \frac{E_1 \cos \delta}{r} \sin q t \\
i &= \frac{E_0}{k\sqrt{1 + \left(\frac{x_0}{r_1}\right)^2}} \cos q_0 t + \frac{E_1 \cos \delta}{r} \sin q t \\
i_0 &= i_1 + i \\
&= \frac{E_0}{\sqrt{r_1^2 + x_0^2}} \{\sin q_0 t + r_1/k \cos q_0 t\}
\end{aligned}\right\} \quad (35)
$$

where:

$$
\left.\begin{aligned}
q_0 &= 2 \pi f_0 \\
x_0 &= q_0 L_0 \\
k &= \frac{1}{q_0 C}
\end{aligned}\right\} \quad (36)
$$

r_1 = resistance of third class conductor.

Substituting into equation (6), gives the terminal voltage e' of the condenser C.

$$
e' = \frac{E_0}{\sqrt{1 + \frac{x_0^2}{r_1^2}}} \sin q_0 t = E_1 \left(\sin \delta + \frac{q L}{r} \cos \delta \right) \cos q t \quad (37)
$$

As seen, the high-frequency term of e' indefinitely increases with decreasing r. That is, the cumulative oscillation builds up to the resonance voltage of L and C, or until limited by the ohmic resistance.

Thus, where r is small, as in the high-voltage coils of power transformers, very high voltages may be produced.

As discussed in *A III* the second term in i_1 and e_1 must always be smaller than the first term. That is, it must be, in i_1:

$$
\frac{E_1 \cos \delta}{r} < \frac{E_0}{\sqrt{r_1^2 + x_0^2}}
$$

Thus we may substitute:

$$
\frac{E_1 \cos \delta}{r} = p \frac{E_0}{\sqrt{r_1^2 + x_0^2}} \quad (38)
$$

where:

$$
p < 1 \quad (39)
$$

Equations (35) thus become:

$$
\left.\begin{aligned}
i_1 &= \frac{E_0}{\sqrt{r_1^2 + x_0^2}} \sin q_0 t \{1 - p \sin q t\} \\
&\text{and:} \\
e_1 &= \frac{E_0}{\sqrt{1 + \frac{x_0^2}{r_1^2}}} \sin q_0 t \left\{ 1 + p \frac{r}{r_1 \cos \delta} \sin (q t - \delta) \right\}
\end{aligned}\right\} \quad (40)
$$

As the second term in e_1 must be less than the first, it must be:

$$
\frac{p r}{r_1 \cos \delta} \leqq 1
$$

or:

$$
r = \leqq \frac{r_0 \cos \delta}{p}
$$

or:

$$
r_1 \cos \delta \geqq p r
$$

Herefrom follows:

(1) With a given resistance r of the oscillating circuit, the oscillation increases, that is, current and voltage build up, until they reach the values, at which the transient effective negative resistance of the third class conductor, $r_1 \cos \delta$, has dropped down to equality with the resistance r, or rather slightly lower.

(2) When in the inductive condenser shunt to a third class conductor, the ohmic resistance r is increased, the cumulative high-frequency oscillation still occurs—though of an amplitude decreasing with increase of the resistance r—until the ohmic resistance becomes equal to the transient negative resistance of the third class conductor, $r_1 \cos \delta$, or rather slightly larger. Then the oscillation stops.

(3) The amplitude e' of the voltage oscillation produced by the third class conductor, is approximately inverse proportional to the resistance of the oscillating circuit. (As seen by equation (37)).

V. Instances

(1) Spark discharge on high-voltage coil of 60,000 volt 3000-kw. 60-cycle power transformer.

Let the constants be:

r = 6 ohms, or about 0.5 per cent resistance of the coil
L = 0.1 h, or about 3 per cent reactance
C = 0.0001 mf.
L_0 = 0.6 h, or about 20 per cent reactance of supply circuit
r_0 = 40 ohms, or about 3⅓ per cent resistance of supply circuit
f_0 = 60 cycles, thus $q_0 = 2 \pi f_0 = 377$
E_0 = 60,000 $\sqrt{2}$ = 85,000 volts.

Assuming $\delta = 60$ deg. as time lag at the oscillation frequency.

It is then, by equations (**35**) to (**37**):

$q = 316{,}000 = 837\, q_0$
$f = 50{,}300$ cycles
$x_0 = 226$ ohms
$q\, L = 31{,}600$ ohms
$k = 26{,}600{,}000$ ohms

Assuming the effective resistance of the oscillating circuit increased to 10 times the ohmic resistance, by the energy losses in the insulation etc. at the high frequency, that is:

$r = 60$ ohms.

and assuming stationary conditions reached by the transient effective resistance of the spark discharge having dropped to equality with the ohmic resistance, that is:

$$r_1 \cos \delta = 60, \text{ and } r_1 = 120 \text{ ohms}$$

this gives:

$\alpha_0 = 62$ deg.

and, denoting:

$\varphi = q_0\, t$, gives:
$e_0 = 85{,}000 \sin (\varphi + 62.)$
$e_1 = 47{,}000 \sin \varphi \{1 + \sin (837\, \varphi - 60.)\}$
$i_1 = 392 \sin \varphi \{1 - \sin 837\, \varphi\}$
$i = 392 \sin \varphi \sin 837\, \varphi$
$i_0 = 392 \sin \varphi$
$e' = 47{,}000 \sin \varphi \{1 - 263 \cos 837\, \varphi\}$

or a maximum value of the condenser voltage of over 12 million volts.

Obviously, no such voltage is reached, but the insulation destroyed before the cumulative oscillation is built up to full value.

(2) Arcing ground on 30 mile three phase transmission line supplied with 30,000 volts, 60 cycles from generator and step up transformer.

Let the constants be:

$r = 4$ ohms
$L = 12$ mh.
$C = 0.3\, \mu$f
$L_0 = 0.2$ h.
$r_0 = 6$ ohms
$f_0 = 60$ cycles, thus $q_0 = 377$
$E_0 = 30{,}000 \sqrt{2} = 42{,}500$ volts.

Assuming $\delta = 45$ deg. as time lag at the oscillation frequency.

It is by equations (**35**) to (**37**):

$q = 16{,}500 = 44\, q_0$
$f = 2640$ cycles
$x_0 = q_0\, L_0 = 75.4$ ohms
$q\, L = 198$ ohms

$$k = \frac{1}{q_0\, C} = 8800 \text{ ohms}$$

Assuming as effective resistance of the oscillating circuit at 2640 cycles, twice the ohmic value, or:

$r = 8$ ohms.

gives as limiting value of the transient effective negative resistance.

$$r_1 \cos \delta = 8, \text{ and } r_1 = 11.3 \text{ ohms}$$

This gives:

$\alpha_0 = 81.5$ deg.

and, denoting:

$\varphi = q_0\, t$; gives
$e_0 = 42{,}500 \sin (\varphi + 81.5 \text{ deg.})$
$e_1 = 6300 \sin \varphi \{1 + \sin (44\, \varphi - 45 \text{ deg.})\}$
$i_1 = 560 \sin \varphi \{1 - \sin 44\, \varphi\}$
$i = 560 \sin \varphi \sin 44\, \varphi$
$i_0 = 560 \sin \varphi$
$e' = 6300 \sin \varphi \{1 - 19.2 \cos 44\, \varphi\}$

or approximately:

$e' = 120{,}000 \sin \varphi \cos 44\, \varphi$

That is, the arcing ground in this case produces a continuous oscillation which builds up to nearly three times the normal line voltage.

It is interesting to note that the cumulative surge of the arcing ground in a transmission line does not reach such excessive values as that in the high-voltage coil of the power transformer, but may be of moderate intensity, of the magnitude of the normal line voltage.