Geometric Solutions of Algebraic Equations

M. Riaz

The American Mathematical Monthly, Vol. 69, No. 7 (Aug. - Sep., 1962), 654-658.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28196208%2F09%2969%3A7%3C654%3AGSOAE%3E2.0.CO%3B2-3

The American Mathematical Monthly is currently published by Mathematical Association of America.

REMOVABLE SINGULARITIES OF ANALYTIC FUNCTIONS

HARLEY FLANDERS, Purdue University

Here is another proof of the standard result:

If $f(z)$ is regular in $0<|z|<R$ and bounded, then there is a unique definition of $f(0)$ which makes f regular on $|z|<R$.

Consider $g(z)=z^2f(z)$. This is continuous on $|z|<R$ with $g(0)=0$, and differentiable at each point of the region. In particular

$$g'(0) = \lim_{z\to 0} g(z)/z = \lim_{z\to 0} zf(z) = 0.$$

Hence g is regular on $|z|<R$ and

$$g(z) = b_2z^2 + b_3z^3 + \cdots$$

is its Taylor expansion, since $g(0)=g'(0)=0$.
Thus $f(z)=b_2+b_3z+\cdots$, which makes the definability of f at 0 crystal clear.

GEOMETRIC SOLUTIONS OF ALGEBRAIC EQUATIONS

M. RIAZ, University of Minnesota

A geometric method for finding the *real* roots of algebraic equations is developed by means of a simple graphical construction made up only of straight lines perpendicular to each other.

The process is derived for the general case of the nth-order algebraic equation which is written in the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0 \qquad (a_n > 0).$$

The set of coefficients $\{a_n, a_{n-1}, \cdots, a_2, a_1, a_0\}$ can be represented geometrically by a corresponding set of directed line segments of length proportional to the magnitude of these coefficients, so juxtaposed that each line segment is drawn at a right angle through the terminus of the preceding segment. The magnitude and direction of a particular line segment (a_k) is given by the vector $a_ke^{j(n-k)\pi/2}$, where $e^{j\pi/2}$ denotes a single 90° turn in the positive or anticlockwise sense taken with respect to the first vector (a_n) used as reference. The resulting geometric figure forms a continuous "nth-order path" with a start O at the origin of segment (a_n) and a finish T at the terminus of segment (a_0). A solution to the equation is determined by finding a new $(n-1)$-order path having the same start O and finish T. The new path progresses as follows: a first line segment drawn through the origin O is terminated at the point where it intersects the infinite line on which the segment (a_{n-1}) is located: at that point, a 90° turn is made defining a second segment which terminates at the intersection with the line determined by segment (a_{n-2}): a further 90° turn is effected. The process is continued until the last intersection occurs at the finish T. One solution of the equation is given by $x=-\tan\theta$, where $\theta<\pm 90°$ is the constant angle between the two paths, counted positive for an anticlockwise angular shift and negative for a clockwise shift. The process is illustrated in Fig. 1 for the case of a 5th-order equation in which all coefficients are positive.

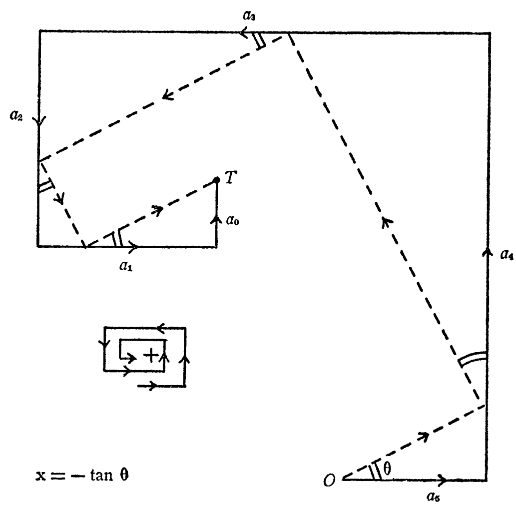

FIG. 1. Geometric solution of the equation

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

(all a's > 0)

That $x=-\tan\theta$ is indeed one of the roots of the algebraic equation may readily be verified by calculating the sequence of lengths corresponding to those sides of the similar rectangular triangles which are opposite to the angle θ:

$$xa_n;\quad x[a_{n-1}+xa_n];\quad x[a_{n-2}+x[a_{n-1}+xa_n]];\cdots;$$
$$x[a_1+x[a_2+\cdots+x[a_{n-2}+x[a_{n-1}+xa_n]]\cdots]]=-a_0,$$

and by observing that the last relation is identical to the original equation, though cast in a different form.

The $(n-1)$-order solution path represents the geometric equivalent of the reduced $(n-1)$-order algebraic equation obtained after extracting the first real root from the original nth-order equation. An $(n-2)$-order solution path taken with respect to the $(n-1)$-order path will in turn produce the second real root and permit a further reduction in the order. The continued reduction process can be followed until all the real roots are determined.

As an example, the cubic equation $x^3-7x-6=0$ is solved geometrically in Fig. 2 through finding a succession of reduction paths which turn out to correspond to the algebraic sequence $x^2-2x-3=0$ (after extracting $x_1=-2$) followed by $x-3=0$ (after extracting $x_2=-1$). Note that even though the coefficient of the x^2 term in the cubic is zero, the first reduction path (a quadratic path) has its first 90° turn located on the line corresponding to the x^2 coefficient of the cubic path. The particular choice of reduction sequence is a matter of convenience.

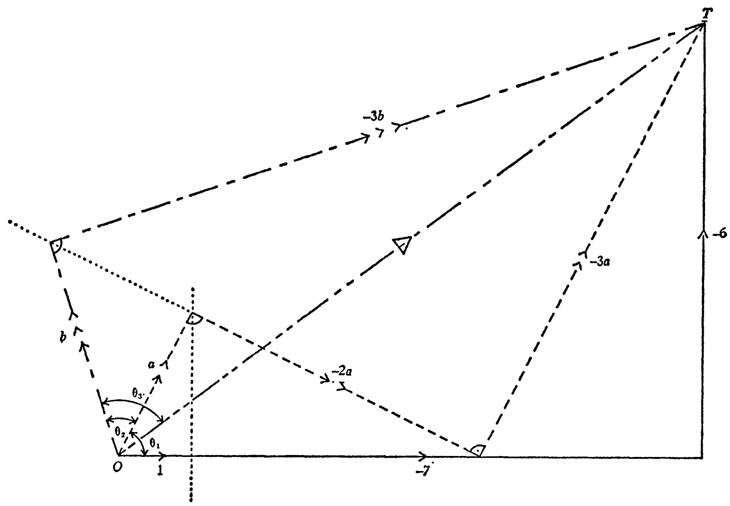

FIG. 2. Solutions of the cubic equation

$$x^3 - 7x - 6 = 0$$

$$x_1 = -\tan\theta_1 = -2$$

$$x_2 = -\tan\theta_2 = -1$$

$$x_3 = -\tan(-\theta_3) = +3$$

It must be pointed out that a reduction procedure as described is not necessary to find the real roots of an algebraic equation. These roots can be obtained directly by determining all the $(n-1)$-order paths that fit the original nth-order path.

The construction of solution paths inherently involves a cut-and-try procedure. However, the simple technique of using a transparent sheet of square millimeter paper can greatly facilitate the search for the solution paths. By turning around the transparent sheet over the drawing of the kth order path, one can readily observe the manner by which the angular shift between the two sheets of paper must be adjusted to meet the constraints of the graphical construction yielding the $(k-1)$-order path. In fact, with a little practice, the adjustment may be effected by scanning visually appropriate lines of the transparency without recourse to actual drawing of construction lines.

The geometrical approach developed here for the solution of algebraic equation may be extended to cover particular situations: for instance, the calculation of the value of an algebraic polynomial for a given value of the variable x, or the calculation of the nth root of a real number. Figures 3 and 4 show the graphical determination of the square and cube roots, and illustrate the degenerate case of a polynomial in which all but the first and last coefficients are zero.

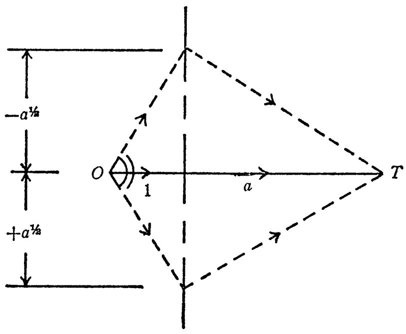

FIG. 3. Square root

$$x^2 - a = 0$$

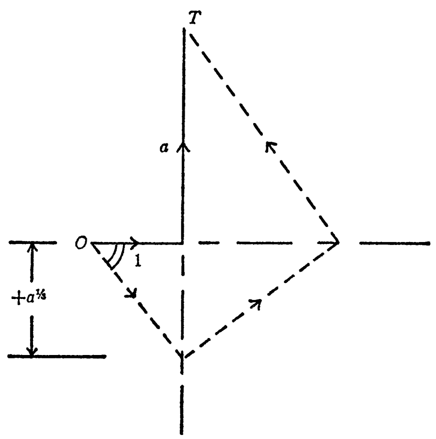

FIG. 4. Cube root

$$x^3 - a = 0$$

A basic advantage to this geometrical type of solution stems from the fact that it provides a ready visual "feel" for the solutions. The effects of changing the coefficients or the order of the original equation on its solution may be estimated rapidly. Certain symmetries in the coefficients of the algebraic form have their geometrical counterparts placed in clear evidence. Being a graphical method, it can only yield the real roots of an algebraic equation with an accuracy that is inherently limited. However, these roots may be calculated to any desired degree of accuracy by following up the graphical method with a numerical method (such as Newton's). By extracting the real roots from the original equation, the order is reduced and the depressed equation may then be solved for its complex roots through applying suitable algebraic or numerical methods.

To emphasize, in conclusion, the main features, this geometric formulation of the solutions of algebraic equations provides:

(a) a conceptually simple method of solution based on purely geometrical considerations,

(b) a simple graphical construction requiring for equipment two triangles, a linear scale and transparent square millimeter paper,

(c) an "analog" technique of solution in contradistinction to the usual "numerical" techniques of algebraic or digital computation,

(d) an isomorphism between algebraic and geometric methods in mathematics.

Reference

1. This graphical technique for the determination of the roots of algebraic equations was first presented by Lill in 1867 ("Nouv. Ann. de math., 2nd series, VI and VII, 1867–68). It is also referred to in a book by Fr. A. Willers entitled Practical Analysis: graphical and numerical methods, Dover Publications, New York, 1948.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D.C.

A DESK—SOME CHAIRS—AND A BLACKBOARD!

ROBERT E. SMITH, Control Data Corporation, Minneapolis

Science, and two modes of thinking. History indicates that many of the basic laws of mathematics came from "laboratory-type" thinking. Early men of mathematics were primarily concerned with the laws of nature, laws of the universe, and the measurement of physical objects. For example, the need for some method of land measurement, by the Egyptians and Chinese, led to basic concepts in geometry. The stars, the moon, the sun, and the earth itself became laboratory objects for the derivation of many other mathematical concepts. Even earlier than this, one can assume that man's possession of five fingers on each hand influenced the basic concepts of arithmetic.

Although one might argue that early mathematical thinking was concerned with that which was *practical*, there is also evidence that an "introspective" or *theoretical* mode of thinking also existed. This is shown by the fact that man puzzled over the characteristics of numbers to the extent that secret cults were sometimes organized around these "discoveries."

History seems to indicate that for many hundreds of years, the two types of mathematical thinking, the "applied" and the "pure," walked hand in hand—each complementing the other. Nevertheless, as scientific knowledge increased, strong forces came into focus which tended to separate the two.