# Impedance Properties of Complementary Multiterminal Planar Structures*

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*Summary*—Booker has shown that Babinet's principle, properly extended to electromagnetic fields, leads to a simple relation between the impedances of two planar complementary structures. A relation, which generalizes this result, is found between the impedance matrices of two complementary $n$-terminal structures.

This relation is applied to the particular $n$-terminal structures having $n$-fold symmetry and to those that are also self-complementary. In the latter case the impedance matrix is real and entirely determined by the number of terminals. It is therefore independent of the exact shape of the elements composing the structure and of the frequency. By connecting in groups the terminals of such a structure various impedance levels, all frequency independent and real, may be achieved.

Structures having their terminal pairs in different locations in the plane are also considered. A self-complementary two-port structure is found to be equivalent, from the impedance point of view, to a length of lossy transmission line having a characteristic impedance of $60\pi$ ohms.

## Introduction

BABINET'S principle, generalized to electromagnetism, gives a relation between the fields scattered or diffracted by two complementary plane structures. The structures to which the principle applies are made of infinitely thin perfectly conducting sheets of arbitrary shape. Two such structures are called complementary if one is obtained from the other by exchanging the open and the conducting portions of the plane.

After obtaining this generalized principle, Booker[1] showed that it implies a precise relation between the impedances of a pair of complementary two-terminal structures measured between closely spaced terminals in the plane of the structure. For example, a narrow slot in a conducting plane, excited across its center, and the complementary narrow strip, forming essentially a dipole with a gap at the center [see Fig. 1(a)], have their impedances $Z_1$ and $Z_2$ related by

$$Z_1 Z_2 = (\tfrac{1}{2}\zeta)^2 \tag{1}$$

where $\zeta$ is the intrinsic impedance of the surrounding medium (in free space, or in air, $\frac{1}{2}\zeta$ is practically $60\pi$ ohms). The same result holds for any pair of complementary two-terminal structures.

The first problem considered in this paper is the generalization of (1) to structures that have more than two terminals. Fig. 1(b) is an example of two complementary three-terminal structures. Since the impedance properties of these structures are now represented by matrices $Z_1$ an $Z_2$ it is readily seen that the relation between them must take a form different from (1).

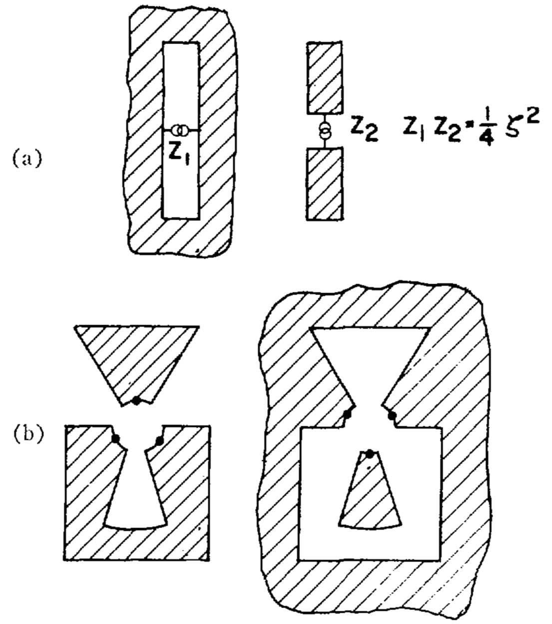


Fig. 1—Two-terminal and three-terminal complementary structures.

Motivation for this investigation came from work done at the University of Illinois on frequency independent antennas. It had been noted by Mushiake and Rumsey[2] that Booker's relation implies that a two-terminal structure congruent to its complement, *i.e.*, "self-complementary," must have an impedance equal to $60\pi$ ohms independent of the frequency. Examples of such structures are shown in Fig. 2. For Fig. 2(a), made up of two right angles, the impedance can be computed[3] directly and is indeed $60\pi$ ohms. The structure in Fig. 2(b) obtained by rotating the arbitrary curve $C$ about $O$ through angles multiple of 90° has also a frequency independent impedance of $60\pi$ ohms.

These considerations may seem of little practical importance since any self-complementary structure contains equal areas of conducting sheets and openings and must therefore be infinite. However, it was found by Rumsey,[2] and DuHamel and Isbell[4] that for some structures of this type fed at the center the currents in the

* This work was supported by Contract AF33(616)-6079, Wright Air Development Center, Dayton, Ohio.
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[1] H. G. Booker, "Slot aerials and their relation to complementary wire aerials," (Babinet's principle), *J. IEE*, pt. III-A, pp. 620–627; March-May, 1946.

[2] V. H. Rumsey, "Frequency independent antennas," 1957 IRE National Convention Record, pt. 1, pp. 114–118.

[3] R. L. Carrel, "The characteristic impedance of two infinite cones of arbitrary cross section," IRE Trans. on Antennas and Propagation, vol. AP-6, pp. 197–201; April, 1958.

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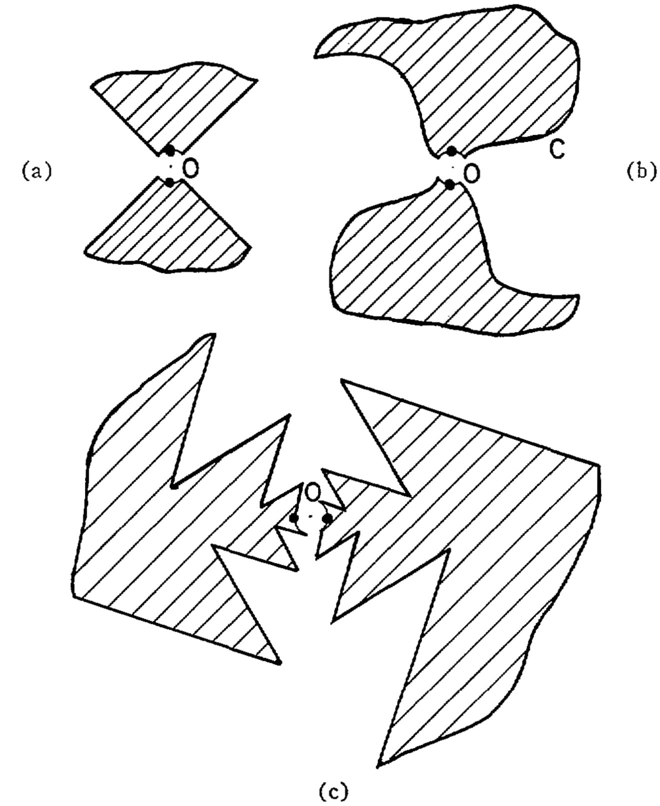


Fig. 2—Self-complementary two-terminal structures.

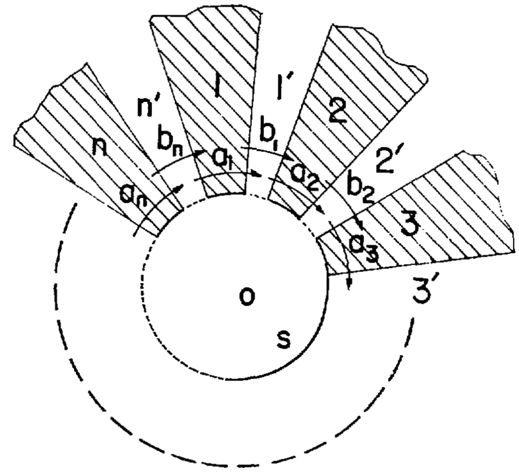


Fig. 3—Feed region of the $n$-terminal structure.

conducting sheets, and the electric fields tangent to the complementary openings, decrease rapidly with distance. The exact law of decrease has not been found but it is faster than $1/r$. As a consequence the structure may be truncated at a finite distance without affecting the input impedance. Fig. 2(c) is an example of such a structure. The impedance measured at $O$ is found to be close to $60\pi$ ohms down to a frequency where the "end effect" becomes noticeable. (This occurs when the distance across the outside teeth approaches half a wavelength.)

Isbell and Mayes considered structures made up of several conducting sectors of this type. By connecting the terminals in groups they were able to obtain frequency independent impedances with values different from $60\pi$.

One purpose of this paper is to show that the measured values can be predicted by using a proper extension of Booker's relation. The extension makes it possible to compute exactly the impedance matrix of a self-complementary $n$-terminal structure having $n$-fold symmetry. This result also gives, through a simple transformation, the characteristic impedance matrix for TEM wave propagation along symmetrical cylindrical and conical structures. Planar structures with terminal pairs at various locations in the plane are also briefly considered.

## Generalization of Booker's Relation

Consider the feed region of an $n$-terminal structure where $n$ conducting sectors come together at a point $O$ (see Fig. 3). Assume that these sectors are limited to the outside of a sphere $S$ small compared to the wavelength of operation. A source located inside $S$ may be connected in several manners to these terminals producing different field configurations about the structure. Since the sphere is small we may describe the various possible connections as we would for a low-frequency circuit without regard for the exact shape of the conducting leads. The situation is completely specified by indicating which groups of terminals are connected to the two terminals of the source. For each such grouping a definite field configuration will result and a definite impedance will be seen by the source.

The method of solution will consist of associating those field configurations produced about complementary structures that are related by duality. From their comparison a relation will result between the corresponding voltages and currents at the terminals and therefore between the impedance matrices of the two structures.

Referring to Fig. 3, consider the various directed paths of integration designated by $a_1, a_2, \cdots a_n, b_1, b_2, \cdots b_n$. They are drawn in the region I above the plane of the structure, very close to it, and have their beginning and end points in the plane. The $b$ paths go from metal to metal and the $a$ paths from opening to opening.

For any path $c$ and any vector field $U$, let us introduce the shorthand notation $c \cdot U$ to indicate the integral of the vector $U$ along the directed path $c$.

$$c \cdot U = \int_c U \cdot ds. \tag{2}$$

Let $F = (E, H)$ be an electromagnetic field produced about the given structure by some configuration of sources inside $S$.

The voltage difference between terminals $i$ and $i+1$ is the integral of the vector $E$ along the path $b_i$

$$V_i - V_{i+1} = b_i \cdot E. \tag{3}$$

The current $I_i$ flowing into terminal $i$ may be expressed by

$$I_i = 2a_i \cdot H. \tag{4}$$

This is seen by noting that the field $F$ has even symmetry with respect to the plane of the structure and that $a_i$ and its image by reflection in the plane (with orientation reversed) form a loop enclosing the $i$th terminal.

The integration paths $a$ and $b$ may be somewhat distorted in the region of the feed point without changing the values of the integrals (3) and (4).

Consider now the complementary structure obtained by replacing the open portions of the plane by conducting plates and replacing the metal by apertures. An acceptable solution for the electromagnetic field about this structure is obtained by taking the dual of $F$ on one side of the plane (Region I for example) and the negative of the dual on the other side (Region II). This gives a field that has even symmetry with respect to the plane and satisfies the new boundary conditions.

The dual of a field $F=(E, H)$ is defined by

$$F' = (E', H') = (-\zeta H, \eta E) \tag{5}$$

where $\zeta$ and $\eta = \zeta^{-1}$ are respectively the intrinsic impedance and the intrinsic admittance of the surrounding space. It is a simple matter to verify that $F'$ satisfies Maxwell's equation when $F$ does.

For the field equal to $F'$ in I and to $-F'$ in II relations similar to (3) and (4) will hold

$$I_i' = 2b_i \cdot H'. \tag{6}$$

$$V_i' - V'_{i+1} = a_{i+1} \cdot E'. \tag{7}$$

They define a set of currents and voltages that may exist at the terminals of the complementary structure and can be produced by a proper arrangement of sources in $S$.

Making use of (5) these may be expressed as

$$I_i' = 2\eta(V_i - V_{i+1}), \tag{8}$$

$$V_i' - V'_{i+1} = \tfrac{1}{2}\zeta I_{i+1}. \tag{9}$$

(By convention in these formulas as well as in (3), (4), (6), (7), $n+1$ is taken as equal to 1.) Formulas (8) and (9) may be collected in matrix form by introducing the vectors $V$, $I$, $V'$, $I'$ having for coordinates $(V_i)$, $(I_i)$, $(V_i')$, $(I_i')$, respectively, and the matrix

$$\Delta = \begin{bmatrix} 1 & 0 & 0 \cdots 0 & -1 \\ -1 & 1 & 0 \cdots 0 & 0 \\ 0 & -1 & 1 \cdots 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & 0 \cdots -1 & 1 \end{bmatrix}. \tag{10}$$

Then (9) and (8) become:

$$\Delta V' = \tfrac{1}{2}\zeta I, \tag{11}$$

$$\Delta^T V = \tfrac{1}{2}\zeta I'. \tag{12}$$

($\Delta^T$ denotes the transpose of $\Delta$.)

These relations are general in the sense that to a condition specified by $V$ and $I$ on one structure is associated another condition described by $V'$ and $I'$ on the complementary structure.

In order to proceed we have to express the relations between $V$ and $I$ and those between $V'$ and $I'$ which describe the properties of the two structures. We shall only consider the case of truly $n$-terminal structures, *i.e.*, those that may be fed arbitrary currents $I=(I_1, I_2, \cdots, I_n)$ with the only restriction that $\sum I_k = 0$. The field configuration about the structure then depends upon $n-1$ independent parameters. When these are given, the voltage difference between any pair of terminals is defined and depends linearly on the vector $I$. Instead of choosing one of the terminals as a zero reference for the voltage it is convenient to use $n$ voltage parameters $V=(V_1, V_2, \cdots, V_n)$ related by the condition $\sum V_k = 0$. An impedance matrix may then be constructed such that

$$V = ZI \tag{13}$$

operates in the $n-1$-dimensional space $P$ defined by the relation

$$\sum_{k=1}^{k=n} X_k = 0 \tag{14}$$

where the $x_k$ are the coordinates of a point.

A similar impedance matrix $Z'$ describes the complementary structure

$$V' = Z'I'. \tag{15}$$

Starting from a given vector $I$ in space $P$ the voltage vector $V$ results from (13). Then the vector $I' = 2\eta\Delta^T V$ represents a set of currents feeding the complementary structure and producing the voltage $V' = Z'I'$. But from relation (11), $I = 2\eta V'$. Finally

$$\Delta Z' \Delta^T Z I = \tfrac{1}{4}\zeta^2 I \tag{16}$$

for any vector in $P$.

This will be expressed by

$$\Delta Z' \Delta^T Z \equiv \tfrac{1}{4}\zeta^2. \tag{17}$$

The sign $\equiv$ is to remind one that the two sides are equivalent only when applied to vectors in $P$. The left hand side transforms any vector into a vector belonging to $P$ and therefore could not be equal to the right hand side without this restriction.

Eq. (17) is the generalization of Booker's relation to $n$-terminal structures. It will now be applied to symmetrical structures and then to self-complementary structures.

## Structures with $n$-Fold Symmetry

Let us now assume that the structure has $n$-fold rotational symmetry. This means that a rotation through the angle $\theta = 2\pi/n$ carries the structure upon itself. For example, Fig. 4 shows two complementary structures

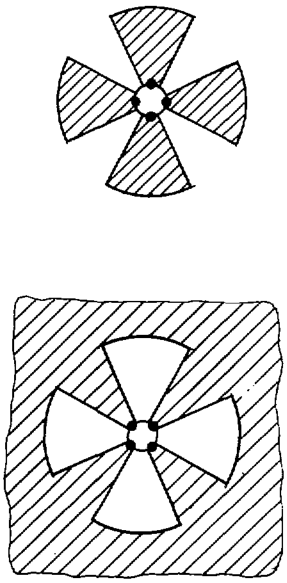

Fig. 4—Two complementary structures having four-fold symmetry.

with 4-fold symmetry. The corresponding matrix $Z$ (and $Z'$ for the complementary structure) will be completely determined by its first row. The next row is obtained by shifting all the elements one step to the right and taking the last element to the first place. The following rows are obtained by the same method and we may write:

$$Z = \begin{bmatrix} Z_0 & Z_1 & Z_2 \cdots Z_{n-1} \\ Z_{n-1} & Z_0 & Z_1 \cdots Z_{n-2} \\ Z_{n-2} & Z_{n-1} & Z_0 \cdots Z_{n-3} \\ \cdots & \cdots & \cdots \\ Z_1 & Z_2 & Z_3 \cdots Z_0 \end{bmatrix}. \tag{18}$$

Because of the symmetry of the matrix $Z$, $Z_{n-1} = Z_1$, $Z_{n-2} = Z_2$, $\cdots$ and the number of parameters is actually $(n+1)/2$ for $n$ odd and $n/2+1$ for $n$ even.

Rather than using matrix notation it is convenient to consider $Z$ as a sequence of $n$ numbers

$$Z = (Z_0 Z_1 \cdots Z_{n-1}).$$

Similarly, $V$ and $I$ are sequences of $n$ numbers

$$V = (V_n V_1 V_2 \cdots V_{n-1}),$$

$$I = (I_n I_1 I_2 \cdots I_{n-1}).$$

(By convention, the index $n$ is equivalent to zero or, more generally, any index is defined modulo $n$.)

The relation between $V$ and $I$ becomes

$$V_i = \sum_k Z_{i-k} I_k \tag{19}$$

and is then expressed as a convolution

$$V = Z * I. \tag{20}$$

In order to represent the product by the matrix $\Delta$ as a convolution we introduce the sequence

$$U = (1\ 0\ 0 \cdots 0)$$

which plays the role of unity for the convolution product ($U * X = X$ for any sequence $X$) and the sequence

$$S = (0\ 1\ 0 \cdots 0).$$

Convolution of any sequence $X$ by $S$ has the effect of shifting each element of $X$ by one step to the right and bringing the last element to the first place.

Multiplication by $\Delta$ then becomes convolution by $U - S$. Introducing also the sequence

$$\check{S} = (0\ 0 \cdots 0\ 1)$$

$\check{S}*$ operates a shift by one step to the left and multiplication by $\Delta^T$ becomes convolution by $U - \check{S}$.

The basic relation (17) between $Z$ and $Z'$ becomes

$$(U - S) * Z' * (U - \check{S}) * Z \equiv \tfrac{1}{4}\zeta^2 \tag{21}$$

or commuting and reducing the factors, making use of the fact that $S * \check{S} = U$,

$$(2U - S - \check{S}) * Z' * Z \equiv \tfrac{1}{4}\zeta^2. \tag{22}$$

[As for (17), this has to hold only when applied to a sequence $I$ such that $\sum I_k = 0$.]

The usual technique for handling an equation of this type is to apply a Fourier transformation which will convert the convolution into an ordinary product. In the case of finite sequences this is also known as finding the "symmetrical components" of the sequence.

Introducing $\epsilon = \exp 2\pi j/n$ and the matrix

$$T = \begin{bmatrix} 1 & 1 & 1 & \cdots 1 \\ 1 & \epsilon & \epsilon^2 & \cdots \epsilon^{n-1} \\ 1 & \epsilon^2 & \epsilon^4 & \cdots \epsilon^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \epsilon^{n-1} & \epsilon^{2(n-1)} & \cdots \epsilon^{(n-1)^2} \end{bmatrix}. \tag{23}$$

The transform of a sequence

$$X = (X_0 X_1 \cdots X_{n-1})$$

is the sequence

$$x = (x_0 x_1 \cdots x_{n-1}) \tag{24}$$

obtained by

$$x = TX$$

where $x$ and $X$ are considered once more as column vectors rather than sequences. The inverse transformation is

$$X = T^{-1}x = \frac{1}{n} T^* x. \tag{25}$$

The asterisk means the complex conjugate. We shall systematically denote the transform of a sequence by the lower case letter corresponding to the capital letter describing the given sequence. Thus the transform of $U$ is the sequence

$$u = (1, 1, \cdots 1)$$

and the transforms of $S$ and $\check{S}$ are, respectively,

$$s = (1, \epsilon, \cdots \epsilon^{n-1})$$
$$s^{-1} = (1, \epsilon^{-1}, \cdots \epsilon^{-(n-1)}).$$

Eq. (22) becomes

$$(2u - s - s^{-1})zz' = \tfrac{1}{4}\zeta^2. \qquad (26)$$

Projecting this relation on the space $P$ simply amounts to neglecting the zero component in the equality. Eq. (26) becomes

$$z_m z_m' \sin^2 \frac{m\pi}{n} = \left(\frac{1}{4}\zeta\right)^2 \qquad (27)$$

for all $m \neq 0$.

This is the complementarity relation for symmetrical structures. It implies that if the impedance properties of a symmetrical structure are known, those of the complementary structure can be determined. Each symmetrical component or eigenvalue of the impedance matrix satisfies a relation similar to the original Booker's relation modified by a factor $\sin^2 m\pi/n$ depending on the order of the component and the number of terminals.

## Self-Complementary Symmetrical Structure

A self-complementary symmetrical structure (with $n$-fold symmetry) may be obtained as shown in Fig. 5. Starting from a curve $C_0$ extending from the origin $O$ to infinity, rotations of $\pi/n$ about $O$ bring it successively in positions $C_0'$, $C_1$, $C_1'$, $C_2$, $\cdots$, $C_{n-1}$, $C_{n-1}'$ ($C_n$ coincides with $C_0$). If the alternate sectors $C_i$ to $C_i'$ are filled with conducting plates, the structure obtained will have $n$-fold symmetry since a rotation of $2\pi/n$ brings it onto itself, and it will be self-complementary since a rotation of half that angle transforms it into the complementary structure.

By choosing for $C_0$ a curve with some oscillations in it or taking a zigzag line as was done by Isbell, a structure is obtained with small end effects. Impedance measurements may be taken on a truncated structure and they should agree with those taken on the infinite structure.

In the relation (27), $z_m' = z_m$, therefore

$$z_m = \frac{\frac{1}{4}\zeta}{\sin\left(\pi \dfrac{m}{n}\right)}, \qquad m \neq 0. \qquad (28)$$

The symmetrical components of the admittance sequence may be taken as

$$y_m = 4\eta \sin \pi \frac{m}{n}, \qquad m \neq 0. \qquad (29)$$

(As noted above we may assume $z_0 = y_0 = 0$ since both the zero order symmetrical components of $V$ and $I$ have been assumed equal to zero.)

By using the inverse transformation (25) the components of $Y$ may be evaluated.

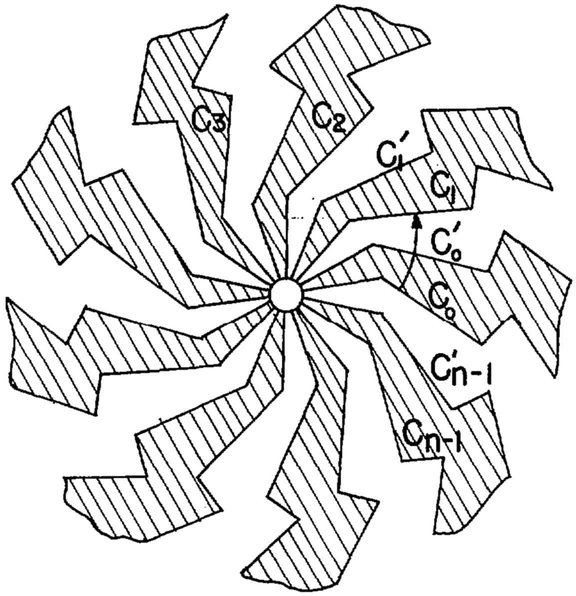


Fig. 5—Self-complementary symmetrical structure (nine-fold symmetry).

After some computation it is found that

$$Y_m = \frac{4\eta}{n} \frac{\sin \dfrac{\theta}{2}}{\cos m\theta - \cos \dfrac{\theta}{2}}, \qquad (30)$$

where $\theta = 2\pi/n$ is the angle of one sector of the structure. Only the coefficient $Y_0$ is positive; all the others are negative and they add up to $-Y_0$ since $y_0 = 0$.

It should be noted that the $z_m$ and $y_m$ are also the eigenvalues of the matrices $Z$ and $Y$ belonging to the eigenvectors $I_{(m)}$ or $V_{(m)}$ represented by the $m$th column of the matrix $T$.

The formula (30) for $Y_m$ has a simple graphical interpretation which may be useful to see how the coefficients of $Y$ vary with $n$ and $m$. If a circle is divided in $n$ equal parts (see Fig. 6 where we have assumed $n=5$) the values of $\cos m\theta$ are read on the $x$ axis as $OM_m$. Considering the point $A$ at angle $\theta/2$ on the circle, the slope of the line $AM_m$ is proportional to $Y_m$ (more exactly the slope equals $-nY_m/4\eta$).

The admittance matrix, or the impedance matrix, of a symmetrical self-complementary structure is entirely defined by the number $n$ of terminals. The coefficients are real and independent of frequency.

## Numerical Results—Experimental Verification

When the admittance sequence ($Y_m$) is known, the impedance properties of any combination of terminals can be computed by simple circuit analysis techniques.

A systematic procedure can be found to deduce first the admittance matrix resulting from a grouping of the $n$ terminals into $p$ sets of connected terminals. If $C = (C_{ij})$ is the $p \times n$ connection matrix defined by $C_{ij} = 1$

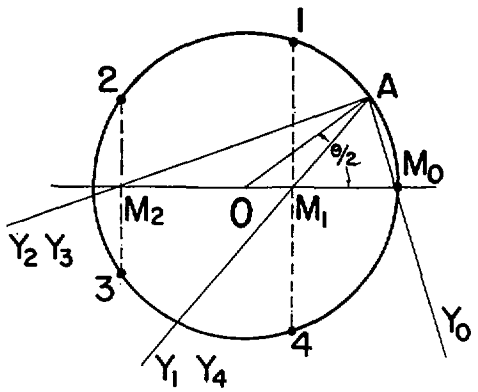


Fig. 6—Construction for the coefficients $Y_{m/\eta}$ (case $n=5$).

when terminal $i$ belongs to set $j$ and $C_{ij}=0$ otherwise, the reduced admittance matrix is

$$Y_C = C^T Y C. \tag{31}$$

If the source is connected between group $j$ and group $k$, all the $I$'s are zero except $I_j$ and $I_k=-I_j$. The voltages are unknown except for the difference $V_j-V_k$. The equations are in sufficient number to define the ratio of $V_j-V_k$ to $I_j$ which is the impedance sought.

The computations have been carried out for a number of configurations involving up to 7 elements and the results compared to experimental measurements.

The measurements are difficult because the feed lines of finite dimensions always disturb the ideal geometry. The thickness of the metal plate is also an important factor. In view of this, the agreement with observed values may be considered as satisfactory.

A plot of measured impedances obtained by D. Isbell and W. Guffey vs computed values (see Fig. 7) leads to the following observations. The experimental values are systematically below the theoretical ones. This may be explained by the finite thickness of the plate and in fact the agreement becomes better for thinner sheets. The percent error for a given thickness increases almost linearly with the number of terminals, independently of the manner in which they are connected.

Disagreement with the theoretical, real, and frequency independent value of the impedance is accompanied by a small variation of the impedance with frequency about a point on the real axis. This variation is of the same order of magnitude as the disagreement. For log-periodic structures the variation is periodic over an approximately circular locus. The values used in Fig. 7 are average impedances corresponding to the center of these circles.

Theoretical values for some of the configurations considered are represented graphically in Fig. 8. It is clear that by increasing the number of terminals, a large range of frequency independent values can, in principle, be obtained.

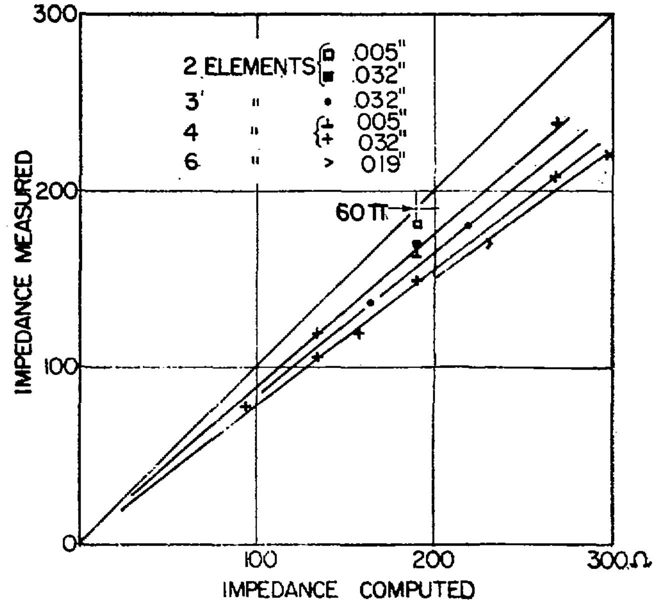


Fig. 7—Comparison with experiment.

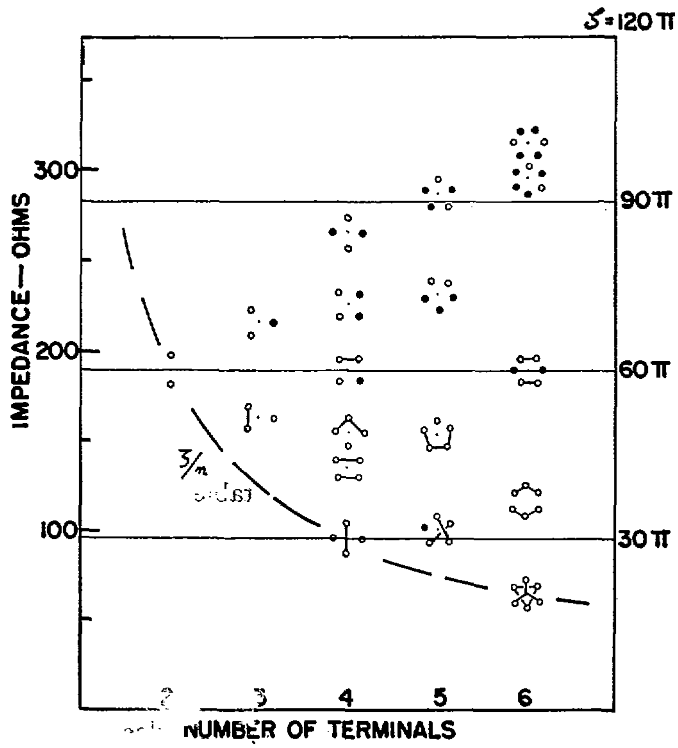


Fig. 8—Table of impedance levels obtained for various configurations of terminals. For each configuration, the two groups of terminals connected to the source are represented by small circles. The floating terminals are represented by black dots.

## Application to Some Electrostatic and TEM Propagation Problems

Eqs. (17), (22), and (24)-(29) have been derived without reference to the particular shape of the elements composing the structure. They do of course apply when these elements are simple angular sectors limited by straight lines [see Fig. 9(b)]. It is known, however,

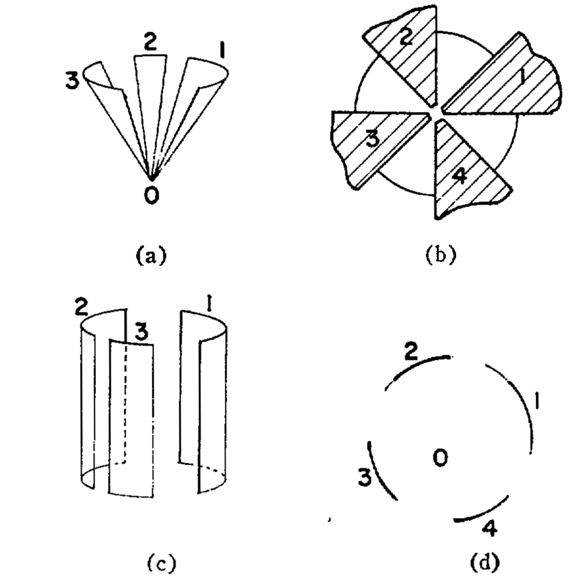


Fig. 9—Related electrostatic and TEM propagation problems.

that these special conical structures support TEM modes of propagation and the admittance matrix of the infinite structure then becomes the characteristic admittance matrix.

Furthermore the characteristic admittance is simply related to the capacitance matrix of the trace of the structure on a sphere of center $O$.

$$\frac{Y}{\eta} = \frac{C}{\epsilon}. \tag{32}$$

The complementary relations (17), (22), and (27)–(29) therefore have their counterparts for the capacitance of structures made of conducting arcs on a circle. For example, a structure made of equal conducting arcs of circles [see Fig. 9(d)] separated by equal openings, has a capacitance matrix representable by a sequence:

$$C_m = \frac{4\epsilon}{\eta} \frac{\sin\frac{\theta}{2}}{\cos m\theta - \cos\frac{\theta}{2}}. \tag{33}$$

(Note that this assumes $\sum Q_m = 0$, hence does not give information about the capacitance of the whole structure connected at a given potential with respect to infinity.)

The circular structure may be placed on an arbitrary sphere and considered as the trace of a conical set of plates [Figure 9(a)]. The characteristic admittance matrix of this conical structure is the same as that of the planar structure from which it comes.[3]

Finally the circular structure may be thought of as the trace of a cylindrical set of plates [Fig. 9(c)] and the characteristic admittance matrix is again the same as for the planar struc[illegible]

The appropriate complementary formulas could have been proved directly for each of these structures but it is worthwhile to note the relations between these problems.

## Structures With Several Terminal Regions

The structures considered so far had all their terminals coming to a point or in practice connected to sources inside a region small in terms of wavelength.

One may also consider structures having terminals at different locations in the plane. Fig. 10 shows an example of a five-terminal structure having two terminal regions. The terminals may be numbered (1, 2, 3) (4, 5). Those of the complementary structure will be (1′, 2′, 3′) (4′, 5′) as shown in the figure. By convention $i+1$ is the terminal "next to" $i$, thus $3+1=1$, $5+1=4$. It is convenient to use as voltage parameter $V_i$, the potential difference between terminal $i$ and $i+1$. The sum of the $V_i$ as well as the sum of $I_i$ is thus zero for every terminal region.

Introducing the shift operator defined by

$$S \begin{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \\ \begin{bmatrix} I_4 \\ I_5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_1 \end{bmatrix} \\ \begin{bmatrix} I_5 \\ I_4 \end{bmatrix} \end{bmatrix}, \tag{34}$$

the relation between the impedance matrices of the two complementary structures becomes

$$Z'Z \equiv \tfrac{1}{4}\zeta^2 S. \tag{35}$$

There again the sign $\equiv$ means that the two sides of the equation give the same result when applied to a vector $I$ such that $\sum I_k = 0$. Eq. (35) is obtained by the same method as (17). This is an alternative form of complementarity which could have been used instead of (17). The only difference is in the choice of the voltage parameters. Those used in (17) were found more convenient in solving the problem of grouping of the terminals.

## Two-Port Self-Complementary Structures

A case of special interest is that of the two-port structure, having two terminal regions with two terminals each.

Choosing at each location, 1 and 2, a +terminal, the currents

$$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{36}$$

are defined as those flowing into $1_+$ and $2_+$, the voltages

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{37}$$

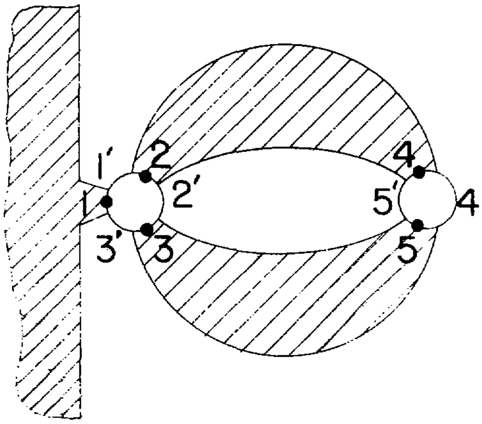


Fig. 10—Planar structure with 3+2 terminals.

are respectively the voltage differences between $1_+$, $1_-$ and $2_+$, $2_-$. The impedance matrix $Z$ relates $V$ and $I$.

$$V = ZI. \tag{38}$$

Fig. 11 shows a self-complementary two-port structure. By reflection into the straight line this structure is transformed into its complement.

Applying the duality transformation to the field as was done in (6)–(9) it is seen that

$$\begin{aligned} V_1' &= \tfrac{1}{2}\zeta I_1, \\ I_1' &= 2\eta V_1, \end{aligned} \tag{39}$$

while

$$\begin{aligned} V_2' &= -\tfrac{1}{2}\zeta I_2, \\ I_2' &= -2\eta V_2. \end{aligned} \tag{40}$$

The sign reversal comes from the fact that in the duality transformation each quantity ($V'$ or $I'$) is related to the dual quantity ($I$ or $V$) belonging to the terminal immediately to its left (seen from the region above the plane). At location 1 in Fig. 11, this relates the two + terminals, but at location 2 it relates the + terminal to the − terminal of the reflected structure.

Introducing a matrix

$$\sigma = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix} \tag{41}$$

the formulas (39) and (40) may be expressed as

$$\begin{aligned} V' &= \tfrac{1}{2}\zeta\sigma I \\ I' &= 2\eta\sigma V, \end{aligned} \tag{42}$$

and using (38) which applies also to $V'$, $I'$,

$$\tfrac{1}{2}\zeta\sigma I = Z 2\eta\sigma V \tag{43}$$

or, finally,

$$(\sigma Z)^2 = \tfrac{1}{4}\zeta^2. \tag{44}$$

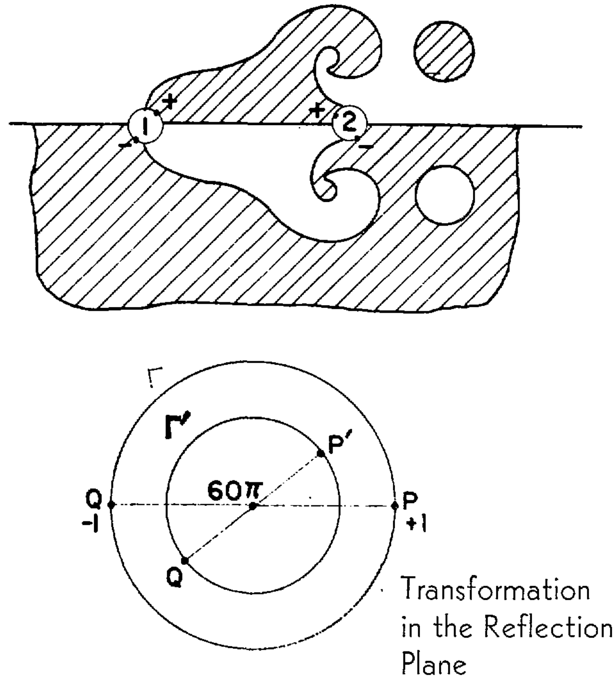


Fig. 11—Self-complementary two-port structure.

This is the relation that $Z$ must satisfy in order to represent the self-complementary structure.

Explicitly, if

$$Z = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}, \tag{45}$$

it follows that

$$\begin{cases} Z_{11} = Z_{22} \\ Z_{11}^2 - Z_{12}^2 = \tfrac{1}{4}\zeta^2. \end{cases} \tag{46}$$

This may be recognized as the impedance matrix of an ideal attenuator having a characteristic impedance of $60\pi$ ohms. The attenuation and phase shift through the element cannot be found from the symmetry considerations but depend on the form of the structure.

Another method of proving the equivalence with an attenuator is to consider the transformation of impedance (or reflection coefficient) through the two-port. If a resistive load of $60\pi$ ohms terminates 2, an impedance of $60\pi$ ohms will be seen at 1. Ploting $60\pi$ at the center $O$ of the reflection chart as in Fig. 11, the point $O$ becomes its own image (iconocenter of the transformation). If an open-circuit load $P$ is mapped at point $P'$, the short circuit load $Q$ will be mapped at $Q'$ corresponding to the reciprocal impedance with respect to $60\pi$. The segment $Q'P'$ has therefore $O$ as its middle point. The image of the unit circle $\Gamma$ is a concentric circle $\Gamma'$. An equivalent circuit for the structure is therefore a length of transmission line with $60\pi$ ohm characteristic impedance in cascade with an ideal attenuator.