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FOREWORD

Mr. Kojima's second book on the abacus gives important information on the further practical use of the abacus and on the principies of its use in business. I believe that his complete explanation of operational methods and their theoretical basis will be of especial help to those foreign students who have no guide or instructor except books.

Aside from its immense utility in business and everyday calculation, the abacus is a far more effective instrument for teaching arithmetic in blind schools than is braille. Moreover, if introduced into ordinary schools, it will prove an excellent time-saver in arithmetic instruction. Half of the problems in arithmetic textbooks are calculation problems and the other half can be reduced to calculation problems by some mathematical reasoning. Consequently, those arithmetic hours allotted for the teaching of abacus operation, by improving the mental arithmetic of students, will enable them to calculate much faster than with pencil and paper, thus creating additional time for a more advanced study of arithmetic.

As Chairman of the Committee of the International Association of Abacus Operators of the Japan Chamber of Commerce and Industry, I have been most pleased to assist Mr. Kojima by making available the findings of recent teclinical and theoretical studies and by revising his manuscript in the light of all the latest information.

Yoemon Yamazaki
Professor of Economics, Nihon University
Vice-President, All-Japan Federation of Abacus Operators
Chairman, Committee of the Int. Assn. of Abacus Operators

I - FURTHER REMARKS ON ABACUS HISTORY

The ancient Chinese books on mathematics which have been preserved furnish hardly any information on the abacus. Accordingly, nothing definite is known about its origin. The only reliable account of the origin of the Oriental abacus is in a book entitled *Mathematical Treatises by the Ancients* compiled by Hsu Yo toward the close of the Later Han dynasty (A.D. 25—220) at the beginning of the third century and annotated by Chen Luan in the sixth century. This book gives sorne information about various reckoning devices of those days and was one of the *Ten Books on Mathematics* (Suan-hwei-shi-chu) which were included among the textbooks to be read for government service examinations in China and Japan for many centuries.

Chen Luan in his note gives the following description of the calculating device:

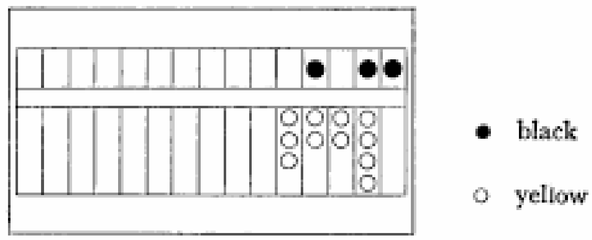


Fig. 1

"The abacus is divided into three sections. In the uppermost and lowest section, idle counters are kept. In the middle section designating the places of numbers, calculation is performed. Each column in the middle section may have five counters, one uppermost five-unit counter and four differently colored one-unit counters."

The above figure represents the abacus as pictured in accordance with the foregoing description. The board represents the number 37 295.

The extent to which the counting board was used may be toid by Hsu Yo's poetical description of the board. The verse, which is highly figurative and difficult to decipher, may read: "It controls the four seasons, and coordinates the three orders, heaven, earth, and man." This means that it was used in astronomical or calendar calculations, in geodetic surveys, and in calculations concerning human affairs.

The reader will notice a close similarity between this original Oriental abacus and the Roman grooved abacus, except for the difference that counters were laid down in the former while they were moved along the grooves in the latter. Because of this and other evidence, many leading Japanese historians of mathematics and the abacus have advanced the theory that the above-mentioned prototype of the abacus was the result of the introduction into the East of the Roman grooved abacus.

The following corroborative pieces of evidence in favor of this theory are cited in the latest works by Prof. Yoemon Yamazaki and Prof. Hisao Suzuki of Nihon University.

(1) The original Chinese abacus has a striking resemblance in construction to the Roman grooved abacus, as is evident in the foregoing quotation from Hsu Yo's book, e.g., four one-unit counters and one five-unit counter in each column.

(2) The method of operation of the ancient Chinese abacus was remarkably similar to the ancient Roman method.

In ancient China, multiplication and division were performed by the repetition of addition and subtraction:

MULTIPLICATION:
Procedure A: 23 x 5 = (23 x 2) + (23 x 2) + 23 = 115 (Ans.)
Procedure B: 23 x 5 = 23 + 23 + 23 + 23 + 23 = 115 (Ans.)

DIVISION:
Procedure A: 115 + 23 = 115 — 23 — 46 — 46 = 0 (Ans.5)
Procedure B: 115 + 23 = 115 — 23 — 23 — 23 — 23 — 23 = 0 (Ans. 5)

In the case of multiplication, each time 23 or 46 was added, 1 or 2 was added to the factor on the left of the board. In the case of division, each time 23 or 46 was subtracted, 1 or 2 was added to the quotient on the left of the board. It is obvious that anyone could easily learn and perform these simple primitive operations.

(3) Traces of reckoning by 5's may be found in the Chinese pictorial representation of reckoning-block calculation as in the Roman numerals, as:

six: VI (5 + 1) seven: VII (5 + 2)
eight: VIII (5 +3) four: IV (5 — 1)

(4) Trade was carried on between China and Rome. Chinese historical documents written in the Han dynasty (206 B.C.-A.D. 220) furnish descriptions of two land routes, called silk roads, connecting the two great empires.

Inasmuch as even in olden days valuable products or devices made in one country were transmitted to others with astonishing rapidity, the above facts may well substantiate this theory.

Among the dozen other reckoning devices mentioned in this book are the reckoning boards pictured below. These boards are presumed to date back to the days of the Chou dynasty, which ended in 249 B.C.

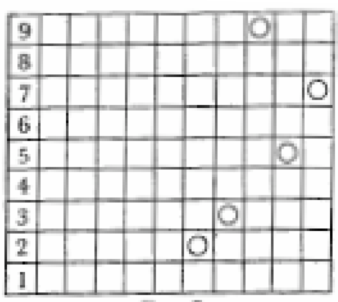


Fig. 2

(The number on the board is 23 957)

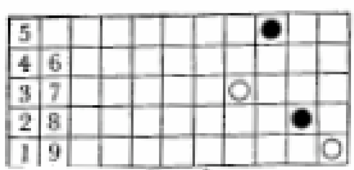


Fig. 3

(When yellow counters were used, the squares in each column represented 1, 2, 3, and 4 respectively. When blue ones were used, they represented 5, 6, 7, 8, and 9 respectively. The black balls in the figure stand for blue counters. The number on the board represents 3581.)

These and other reckoning devices are believed to have gone out of use as the previously mentioned abacus developed and gained popularity.

Now two questions present themselves. One is why this ancient abacus developed in the East to become such an efficient calculating machine. The other is why this development did not also take place in the West.

The reasons lay perhaps in the systems of calculation and the numerical nomenclature which were used in the East and West. They differ significantly. In ancient China and Japan numbers were named, written, and set on a calculating board from left to right, from the highest denomination to the lowest. Thus the introduction of the abacus to China provided the Chinese with an ideal tool in terms of their method of naming and using numbers. This compatibility and normal inventiveness caused the primitive abacus to be developed into its modern form during the long development of the Chinese civilization.

The chief calculating devices which are known to have been used in China from before 1 000 B.C. to the days when the abacus came into wide use are reckoning blocks called *ch'eou* in China and *sangi* in Japan and slender bamboo sticks called *chanchu* in China and *zeichiku* in Japan. The former device continued to be used in the East for calculation until not many years ago, and the latter device, which was more awkward, was largely replaced by the former for calculating purposes and is presently used only by fortunetellers for purposes of divination.

Until the introduction of Western mathematics, mathematicians in China and Japan utilized reckoning-block calculation, which had not only been developed to the point of performing basic arithmetic operations but was also used to solve quadratic, cubic, and even simultaneous equations. It is presumed that they did not think it worth while to concern themselves with the other reckoning devices, including the abacus, which was, in their eyes, an inferior calculator barely capable of performing multiplication and division by means of the primitive cumulative method of addition and subtraction. Probably another reason which alienated mathematicians from these reckoning devices was that these instruments gave only the result of calculation, and were incapable of showing either the process of calculation or the original problem.

In ancient times China was primarily a nomadic and agricultural country, and business in those days liad little need of instruments of rapid calculation. Anyway a millennium after the Han dynasty there was no record of the abacus. During the dozen centuries beginning with its first mention in the Han dynasty until its development, this primitive calculator remained in the background.

However, with the gradual rise of commerce and industry, the need for rapid calculation grew. The modern, highly efficient abacus, which probably appeared late in the Sung dynasty (906—1279), came into common use in the fourteenth century. The great rise and prosperity of free commerce and industry during the Ming dynasty (1368 - 1636) are presumed to have promoted the use and development of the abacus. A number of books on mathematics brought out in those days give descriptions of the modern Chinese abacus and give accounts of the modern methods of abacus operation, including those of multiplication and division.

Bamboo, indigenous in the East, has furnished an abundant source of ideal material for an efficient and inexpensive abacus. Since the Ming period, on account of its remarkable efficiency, low price, and handiness, the abacus has been the favorite instrument of calculation in the East.

The Chinese abacus of the Ming period had two five-unit counters and five one-unit ones on each rod. The primitive abacus was changed into the present Chinese form to suit the convenience of figuring up the Chinese weights not based on the decimal system. The weights were also important for conversion of currency. Another cogent reason why the Chinese abacus has two five-unit counters on each rod is that a rod with two five-unit counters is more convenient to abacus operation by means of the Chinese method of multiplication and also by means of the older method of division which uses a special division table.

In Europe, the line abacus or counting board appeared first in France about the beginning of the thirteenth century and rapidly became popular. From the fourteenth to the seventeenth century the practice of this manual arithmetic was universal in business and in households, as well as in the departments of government. Its immense popularity may well be illustrated by the following pleasantly expressed stanza attributed to Brébeuf as it is quoted in Francis Pierrepont Barnard's *Casting Counter and the Counting-Board.*

Les courtisans sont des jetons;
Leur valeur dépend de leur place;
Dans la faveur, des millions;
Et des zéros dans la disgrâce.

The same book also quotes the phrase, "Faux comme un jeton," which arose from the practice of gilding or plating jettons and passing them as money, or creating a deceptive impression.

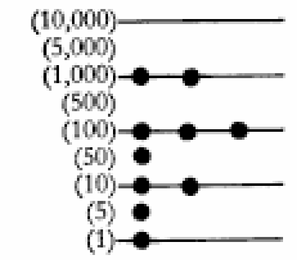


Fig. 4

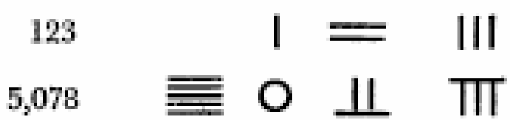
The number 2 376 would be expressed by jettons on the line abacus or counting board as in Fig. 4.

(Each line upwards is ten times the value of that below it. Each space is five times as much as the line next below it. In addition, the process began at the units, and in subtraction at the higher digits.)

However, in Europe the une abacus failed to develop into the efficient rod abacus, and gradually gave way to the cipher system of greater efficiency, until it was given the *coup de grâce* by the French Revolution, which enforced the nation-wide ciphering system. One of the major causes for this result is presumed to be found in the fact that before the introduction of Arabic numerals European countries used diverse systems of numerical notation-duodecimal, binary, sexagesimal, etc. The division of daytime into twelve hours and that of one hour into sixty minutes, etc. may be mentioned as vestiges of these numerical systems. The rod abacus can never be worked with efficiency on these numerical scales. Another remote cause may be traced back to the way in which the Arabs, who introduced the cipher system into Europe, named their numbers. The Semites, including the Arabs, named their numbers beginning at the units, although they wrote from right to left. Thus for instance, in Arabic, one hundred and twenty-five was called five and twenty and one hundred, the result appearing as 125, as Prof. Cargill G. Knott says in his treatise on the abacus. This is believed to be the primary reason why the Arabs, who achieved remarkable development in mathematics in the medieval ages, made use of their Arabic numerals without recourse to the less efficient lime abacus or other calculating devices. Nor could the rod abacus have been used with efficiency by a race which used such a system of naming their numbers. The early Indians, who are credited with the invention of the cipher, spoke like the Arabs although they wrote from left to right. The Chinese named their numbers beginning with the largest denomination, although they wrote from top to bottom, proceeding from right to left.

Now the second question before us is: What causes prevented the adoption of a cipher system in China and Japan? The Chinese and the Japanese write in vertical columns from top to bottom, while the cipher system is worked from left to right. However, this is not considered the primary cause, for in the remote past coeval with the origination of Chinese characters, the Chinese carried out their calculation by means of reckoning blocks working left to right. However, this reckoning-block calculation was cumbersome and was no more fit for rapid operation than the Western line abacus.

A couple of examples of the arrangement of reckoning blocks are given below. The numbers 123 and 5 078 are represented:



In the units, hundreds, and other odd places, the numbers up to five are each represented by the corresponding number of vertical strokes, and the numbers from six to nine are each represented by the addition of the requisite number of strokes below a five-unit horizontal line. In the tens, thousands, and other even places, the numbers up to five are each represented by the corresponding number of horizontal strokes, and the numbers from six to fine are each represented by the addition of the requisite number of strokes aboye a five-unit horizontal line.

The Chinese numerical notation, which was probably the pictorial representation of reckoning-block calculation, was of far less practical use in calculation than reckoning blocks. Accordingly, mathematical calculation was generally performed with reckoning blocks and later also with the abacus. In remote antiquity, probably more than 2 000 years back, reckoning blocks were arranged differently for calculation. In those days the numbers in units and hundreds places were represented by horizontal blocks instead of vertical ones, and numbers in tens and thousands places were represented by vertical blocks. Thus the Chinese numerals, 一 (1), 二(2), 三 (3), and 百 (100), comprised of horizontal strokes, are pictographs, representing horizontally arranged blocks (- = ≡), and the numerals, 十(10), 廿(20), 卅(30), and 千 (1 000), comprised of vertical strokes, are pictographic imitations of blocks arranged vertically.

What caused the change in the arrangement of reckoning blocks is a knotty problem, to which no satisfactory solution has been offered. However, some scholars conjecture that because of the great importance of divination in early China the arrangement of reckoning blocks might probably have been influenced by the method of arranging reckoning sticks for divining purposes.

Probably the major cause which prevented the replacement of the cumbersome numerical notation by the cipher system was the development of the abacus, which could meet everyday public needs in business and household calculation. Reckoning-block calculation, which was applied to the primitive abacus, made a remarkable development, and by the Ming dynasty the abacus had become a far more efficient computer than the cipher system.

In the days of feudal government, learning was mostly of classics and was the exclusive heritage of certain officials and limited circles of scholars. Mathematics was studied only by the few who were initiated into this mystery of learning, and many of them formed exclusive esoteric sects of hereditary transmission to preserve their patrimonial positions or living. Under these social conditions it was none of their concern to teach or popularize their secrets of mathematics. The enlightened scholars who were favored with exceptional opportunities to study Western science and mathematics may have been aware of the superiority of Arabic numerals to the cumbersome Oriental numerical notation. But these intellectuals must have been too few and far between and their outcry to initiate the reform too feeble to arouse public attention.

Among the other important causes may be mentioned the want of free international trade and communication, the virtual isolation of Eastern countries from the West, and the consequent lack of understanding of international situations and national prejudice against foreign culture, and among the rest, the conservatism of human nature. The Chinese officialdom was so prominently conservative that it would firmly have resisted any attempts at such reforms or improvements in the hoary customs or timehonored classics of national veneration, many of which had been included among textbooks for government service examinations during the long Chinese historical period extending over twenty centuries.

In Japan it was not until several years after the 1868 political revolution, which overthrew the shogunate (government by the supreme feudal ruler), that the progressive modem government, awakened to the progress of the world, enacted the compulsory education law, including in the curriculum the cipher system, without which the effective teaching of modern mathematics to the public is impossible.

Now the Japanese word for abacus, *soro ban,* is probably the Japanese rendering of the Chinese *suan-pan, (soo-pan* in the southern dialect or *sur-pan* in Manchuria). The soroban in Japan did not come into common use until the seventeenth century. However, the historical fact that beginning with the seventh century, there were at times as many as 2 000 Japanese students studying at the then Chinese capital in Chang-an, now called Si-an, furnishes us with reliable evidence that the abacus was introduced into Japan at a far earlier date, although the oldest documentary evidence of the Japanese abacus does not date further back than the sixteenth century.

In any case, once this convenient instrument of calculation gained popularity in Japan, it was studied extensively and intensively by many mathematicians including Seki Kowa (1640—1709), who discovered a native calculus independent of the Newtonian theory. As a result, the form and methods of operating the abacus have undergone one improvement after another. For a long time in Japan two kinds of abacus were used concurrently until the 1868 political revolution: the Chinese-style one with two five-unit counters and five one-unit counters and the older Japanese-style one with one five-unit counter and five oneunit counters. After the time of the revolution, the Chinese-style abacus went completely out of use. Finally since around 1940, the older-style Japanese abacus has largely been replaced by the present more advanced and efficient one with one five-unit counter and four one-unit counters.

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